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EXISTENCE OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STEPANOV FORCING TERMS.

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Abstract. We introduce a new concept of Stepanov weighted pseudo almost periodic functions of class r which have been established by recently in [20]. Furthermore, we study the uniqueness and existence of Stepanov weighted pseudo almost periodic mild solutions of partial neutral functional differential equations having the Stepanov pseudo almost periodic forcing terms on finite delay.

1. Introduction

It is well known that Periodicity is natural and important phenomena in the real world. However real systems usually exhibit internal variations or external perturbations which are only approximately periodic. The theory of almost periodic functions was introduced in the literature around 1924-1926 with the pioneering work the Denish mathematician Harald Bohr. Many authors have furthermore generalized in different directions the notion of almost periodicity for more realistic decription to real world phenomenon around us. A various types of Almost periodic systems describe world more realistically than periodic ones. Since then, this concept has undergone several interesting, natural, and powerful generalization, such as pseudo almost periodicity, weighted pseudo almost periodicity, weighted pseudo periodic of class r and so on. We refer to [3, 5, 10, 16] and references cited therein for more details on the subject. Diagana [6] introduced the concept of Stepanov-like weighted pseudo almost periodicity and applied the concept to study the existence and uniqueness of weighted pseudo almost periodic solutions.

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In [7], Diagana investigate some sufficient conditions for the existence and uniqueness of pseudo almost periodic mild solutions to the class of abstract partial evolution equations given by

$$\frac{d}{dt}[u(t) + f(t, Bu(t))] = Au(t) + g(t, Cu(t)),\ t \in \mathbb{R}$$

where A is the infinitesimal generator of an exponentially stable semigroup acting on $\mathbb{X}, B, C$ are arbitrary densely defined closed linear operators on $\mathbb{X}$ and f, g are some jointly continuous functions. They obtained the existence and uniqueness of a pseudo periodic mild solution under the some appropriate assumptions. Further, their main results may serve to characterize pseudo almost periodic solutions to partial neutral functional differential equations [5], integro-differential equations [2],[4],[11] and others.

Xia [18] introduce the notion of weighted pseudo S-asymptotic periodicity in the Stepanov sense, which generalizes the known concepts in different directions and worked the existence and uniqueness of weighted pseudo S-asymptotically ω-periodic solutions for the following equation

$$\frac{d}{dt}[u(t) + f(t, u(t))] \quad = \quad Au(t) + g(t, u(t)), t \in \mathbb{R}, , u(0) = u_0,\ t \in \mathbb{R}^+,$$

where ω is an integer.

In recent years, Zheng in [20] worked a further investigation on the composition results for weighted Stepanov-like pseudo periodic functions of class r and proved a new composition theorem for weighted Stepanov-like pseudo periodic functions of class r. And then, they prove the existence and uniqueness of weighted periodic solution to the following semi-linear delay differential equaton with a weighted Stepanov-like pseudo periodic nonlinear term

$$u'(t) \quad = \quad Au(t) + f(t, u_t), t \in R,$$

where ω is an integer. Motivated by the previous works, we investigate that the existence and uniqueness of Stepanov weighted pseudo almost periodic mild solutions for the following system having the Stepanov pseudo almost periodic forcing terms on Banach space.

$$\frac{d}{dt}[u(t) + f(t, Bu_t)] \quad = \quad Au(t) + g(t, Cu_t), t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup acting on X, $u_t \in \mathfrak{B}$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$, r is nonnegative constant, f, $\mathfrak{B}$ are specified in the later and B,C are arbitrary densely defined closed linear operator on $\mathbb{X}$, and f, g are some jointly continuous functions. an appropriate function that will be given later.

The organization and main ideas of this article are briefly described as follows. In the second section, we recall some definitions and results related with Stepanov almost periodic functions. In third section, after providing some lemmas and preliminary results which will be used through this paper. We give a main results, inspired from the previous papers. In forth section, we give an example to illustrate our main results.

2. Preliminaries and Notations

For a given $T > 0$ and each $\rho \in U$, set $\mu(T,\rho)= \int_{-T}^{T}\rho(t)dt$.
In order to facilitate our discussion, we introduce the following notations:

$$\begin{array}{rcl} U & : & \{\rho \in L^1_{loc}(\mathbb{R}) : \rho(t) > 0,\ a.e.t \in \mathbb{R}\} \\ U_\infty & : & \{\rho \in U : \lim\limits_{T\to\infty} \mu(T,\rho) = \infty\} \\ U_B & : & \{\rho \in U_\infty : \rho \text{ is bounded}, \inf\limits_{x\in R} \rho(x) > 0. \end{array}$$

For a Banach space $(X, ||\cdot||)$ and $(Y, ||\cdot||)$,

$$\begin{array}{rcl} BC(\mathbb{R},X) & : & \{f : \mathbb{R} \to X : \text{ the Banach space of bounded continuous functions}\} \\ L^p(\mathbb{R},X) & : & \{f : \mathbb{R} \to X : \text{ the space of all classes of equivalence} \\ & : & \text{of measurable function such that } ||f|| \in L^P(\mathbb{R},\mathbb{R})\} \\ L^p_{loc}(\mathbb{R},X) & : & \{f : \mathbb{R} \to X : \text{ the space of all classes of equivalence} \\ & : & \text{of measurable function such that the restriction of } f \\ & : & \text{to every bounded subinterval of R is in } L^P(\mathbb{R},\mathbb{R}^+)\} \\ \mathfrak{B} & : & \text{The space } C([-r,0],X) \text{ endowed with the sup norm } ||\psi||_{\mathfrak{B}} \text{ on } [-r,0]. \end{array}$$

Next, we review some definitions and lemmas well known from our references ([5, 12, 15, 17, 19]).

Definition 2.1. A function $f \in C_b(\mathbb{R},X)$ is called *almost periodic* if for every $\epsilon > 0$, if there exists an l such that every interval of length $l(\epsilon)$ contains a number τ with property that

$$||f(t+\tau) - f(t)|| < \epsilon,\ \text{for every}\, t \in \mathbb{R}.$$

The collection of such functions is denoted by $AP(\mathbb{R}, X)$.

We need to define new notions for the delayed equations for which the history belong to $C([-r,0];X)$.

DEFINITION 2.2. A function $f \in BC(\mathbb{R}, X)$ is said to be a *weighted-pseudo ergodic of class r* if

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-\tau,t]} ||\phi(\theta)|| \Big) \rho_2(t)dt = 0.$$

The collection of such functions is denoted by $WPAP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$.

DEFINITION 2.3. Let $\rho_1, \rho_2 \in U_\infty$. A continuous function $f \in BC(\mathbb{R}, X)$ is called *weighted pseudo almost periodic of class r*
if it can be written as

$$f = h + \varphi,$$

with $h \in AP(X)$ and $\varphi \in WPAP_0(\mathbb{R}, X, \rho_1, \rho_2)$.
The collection of such functions is denoted by $WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Let $||\cdot||$ denote the norm of space $L^p(0,1;X)$ for $p \in [1,\infty)$, we need the definition as followings.

DEFINITION 2.4. The bochner transform $f^b(t,s),\ t \in \mathbb{R},\ s \in [0,1]$ of a function $f : \mathbb{R} \to X$ is defined by

$$f^b(t,s) = f(t+s).$$

The bochner transform $f^b(t,s,u),\ t \in \mathbb{R},\ s \in [0,1],\ u \in Y$ of a function $f : \mathbb{R} \times Y \to X$ is defined by

$$f^b(t,s,u) = f(t+s,u).$$

DEFINITION 2.5. Stepanov norm :
For a positive number L and $f, g \in L^1_{loc}(\mathbb{R}, X)$,

$$||f||_{S^p_L} = \sup_{x\in R} \Big[\frac{1}{L}\int_x^{x+L} ||f(t)||^p dt\Big]^{\frac{1}{p}}, p \geq 1.$$

DEFINITION 2.6. Let $p \in [1,\infty)$. The space $BS^p(\mathbb{R}, X)((\mathbb{R} \times Y, X))$ of all Stepanov bounded functions, with the exponent p, contains of all mesurable functions f on $\mathbb{R}$ with values in X such that $f^b \in L^\infty(\mathbb{R}, L^p((0,1);X))$. This is a Banach space with the norm

$$||f||_{S^p} = ||f^b||_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in R} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big)^{\frac{1}{p}} = \sup_{t\in R} ||f(t+\cdot)||_p.$$

We need the following weighted Stepanov ergodic space in $BS^p(\mathbb{R}, X)$:

Definition 2.7. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be a *S^p-weighted ergodic* if

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} ||f(s)||^p ds \Big)^{\frac{1}{p}} dt = 0.$$

The collection of such functions is denoted by $S^pWPAP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Definition 2.8. [19] A function $f \in BS^p(\mathbb{R}; X)$ is said to be *weighted pseudo almost periodic in the sense of Stepanov(S^p-weighted pseudo periodic) of class r* if we want to highlight the degree p if it can be written as

$$f = h + \varphi,$$

with

$$h \in AP(X) \text{ and } \varphi \in S^pWPAP_0(\mathbb{R}, r, X, \rho_1, \rho_2).$$

We denote the set of all such functions by $S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Under the suitably condition, the decomposition is unique. Similar proofs and contents are well explained and can be easily proved in [20].

Theorem 2.9. *Let $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} > 0$. Then the decomposition of weighted Stepanov pseudo almost periodic function of class r is unique.*

For our main results, we introduce main composition theorem for weighted Stepanov pseudo periodic function which have been established recently in [20]. A composition theorem is a very important when it comes to dealing with Stepanov weighted pseudo almost periodic evolutions equations. Shan [20] establish composition results for weighted Stepanov pseudo periodic function with following hypothese:

(H_1) For any $\epsilon > 0$, there exists $\sigma > 0$ such that $x, y \in L^P(0,1;X)$ and $||x-y||_p < \sigma$ imply that

$$||h(t+\cdot, x(\cdot)) - h(t+\cdot, y(\cdot))||_p < \epsilon,\ t \in \mathbb{R}.$$

Theorem 2.10. [20] *Let $\rho_1, \rho_2 \in U_\infty$, $r \geq 0$, $f = g+\phi \in S^pWPAP(\mathbb{R}\times X, X, r, \rho_1, \rho_2)$, $h = h_1 + h_2 \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ with $\overline{h_1(\mathbb{R})}$ compact, $g(t+\omega) - g(t) = 0$, $h_1(t+\omega) - h_1(t) = 0$. Assume g satisfies (H_1), ϕ satisfies (H_1) and $\{f(\cdot, z) : z \in K\}$ is bounded in $S^pWPAP(\mathbb{R}\times X, X, r, \rho_1, \rho_2)$ for any bounded $K \subset X$, then $f(\cdot, h(\cdot)) \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$.*

To study the system for delay we need the following Lemma:

Lemma 2.11. [20] *Let $\rho_1, \rho_2 \in U_T$, $u \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$, then u_t belongs to $S^pWPAP(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$.*

Corollary 2.12. [20] *Let* $\rho_1, \rho_2 \in U_T,\ u \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$, *then* u_t *belongs to* $WPAP(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$.

Motivated from work of [7], [12] we define the mild solution of Stepanov weighted pseudo almost periodic equation as following.

Definition 2.13. Let $T(t)$ be the C_0-semigroup generated by A and $g \in L^1(\mathbb{R}, X)$. The function $u(t) \in C(\mathbb{R}, X)$ given by

$$u(t) \quad = \quad -f(t, Bu_t) - \int_{-\infty}^{t} AT(t-s)f(s, Bu_s) + \int_{-\infty}^{t} T(t-s)g(s, Cu_s)ds,$$

is the mild solution of equation (1.1) provided that the function $s \to AT(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for $t \in \mathbb{R}$.

3. Existence results for Stepanov weighted pseudo almost periodic mild solution

To work (1.1) we established some sufficient criteria as following:

(H_2) Let $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} > 0$ and $\sup_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} < \infty$.

(H_3) The linear operator $B, C \in B(Y, X)$ with $\max\left(||B||_{B(Y,X)}, ||C||_{B(Y,X)}\right) = \bar{\omega}$.

(H_4) The operator $A : D(A) \subset X \to X$ is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ such that there exist constants $M > 0$ and $\delta > 0$ with

$$||T(t)|| \leq Me^{-\delta t} \text{ for all } t \geq 0.$$

Moreover, the function $\sigma \to AT(\sigma)$ defined from $\mathbb{R}^+$ into $B(Y)$ is strong (Lebesgue) measure and there exists a nonincreasing function $\gamma : \mathbb{R} \to [0, \infty)$ and a constant $\omega > 0$ with $\eta = \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} e^{-\omega q s} ds\right)^{\frac{1}{q}} < \infty$ such that

$$||AT(t)||_{B(Y)} \leq e^{-\omega t} \cdot \gamma(t),\ t \in \mathbb{R}^+.$$

(H_5) $f, g \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ and there exist constant $L_f, L_g > 0$ such that

$$||f(t, \psi_1) - f(t, \psi_2))||_p \leq L_f ||\psi_1 - \psi_2||_{\mathfrak{B}},$$

and

$$||g(t, \psi_1) - g(t, \psi_2))||_p \leq L_g ||\psi_1 - \psi_2||_{\mathfrak{B}}$$

for all $\psi_i \in \mathfrak{B},\ i = 1, 2$ and $t \in \mathbb{R}$.

(H_6) Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote

$$\alpha = M\Big(\frac{e^{qc-1}}{qc}\Big)^{\frac{1}{q}} \sum_{k=1}^{\infty} e^{-ck}.$$

LEMMA 3.1. *We assume that the hypothesis $(H_2), (H_4)$ is satisfied. If $f \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2),\ \rho_1, \rho_2 \in U_\infty$ then $(\Gamma_1 f)(t) = \int_{-\infty}^{t} AT(t-s)f(s) \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$.*

Proof. For $n \leq t \leq n+1,\ n \in \mathbb{N}$, we have

$$\begin{aligned}
||(\Gamma_1 f)(t)|| &\leq ||\int_{-\infty}^{t} AT(t-s)f(s)ds|| \\
&\leq \int_{-\infty}^{t} Me^{-\delta(t-s)}||f(s)|| \\
&\leq \sum_{k=1}^{n} \int_{t-k}^{t-k+1} Me^{-\delta(t-s}||f(s)||ds \\
&= \sum_{k=1}^{n} Me^{-\delta(n-t+k-1}\Big(\int_{t-k}^{t-k+1} ||f(s)||^p ds\Big)^{\frac{1}{p}} \\
&= L||f||_{S^p}.
\end{aligned}$$

This means that Γ_1 is bounded.
Next we show that Γ_1 is continuous.

$$\begin{aligned}
||(\Gamma_1 f)(t+\epsilon) - (\Gamma_1 f)(t)|| &\leq ||\int_{-\infty}^{t+\epsilon} AT(t+\epsilon-s)f(s)ds - \int_{-\infty}^{t} AT(t-s)f(s)ds|| \\
&\leq \int_{-\infty}^{t} AT(t-s)||\big(f(s+\epsilon) + f(s)\big)||ds \\
&\leq \sum_{n=0}^{\infty} \int_{t-n+1}^{t-n} AT(t-s)||f(s)||ds \\
&\leq M\sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\delta(t-s)}||f(s)||ds \\
&\leq M\sum_{k=1}^{\infty} \Big(\int_{t-k}^{t-k+1} e^{-\delta q(t-s)}ds\Big)^{\frac{1}{q}} \Big(\int_{t-k}^{t-k+1} ||L(s)||^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\frac{\delta q}{M^q}\Big)^{\frac{1}{q}} \Big(\frac{e^\delta - 1}{e^\delta}\Big).
\end{aligned}$$

It is clear that Γ_1 is coninuous.
Finally we must show that

$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}||\psi(\theta)||\Big)dt=0,$$

where $\psi(s)=\int_{-\infty}^{t}AT(t-s)f(s)ds$.
For some constant $\tilde{M}$, we have

$$\begin{aligned}
||(\psi(s)|| &\leq ||\int_{-\infty}^{t}AT(t-s)f(s)ds|| \\
&\leq \int_{-\infty}^{t}Me^{-\delta(t-s)}||f(s)|| \\
&\leq \sum_{k=1}^{n}\int_{t-k}^{t-k+1}Me^{-\delta(t-s}||f(s)||ds \\
&= \sum_{k=1}^{n}Me^{-\delta(n-t+k-1}\Big(\int_{t-k}^{t-k+1}||f(s)||^p ds\Big)^{\frac{1}{p}} \\
&= \tilde{M}\Big(\int_{t-k}^{t-k+1}||f(s)||^p ds\Big)^{\frac{1}{p}}.
\end{aligned}$$

In the mean time, since $f\in S^pWPAP(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$, there exists $m\in\mathbb{N}$, such that

$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\big(\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}||f(s)||^p ds\big)^{\frac{1}{p}}\Big)dt<\epsilon \text{ for } T\geq m$$

and let

$$M(T,\epsilon,f)=\{t\in[-T,T]:\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{\theta+1}||f(s)||^p ds\Big)^{\frac{1}{p}}\geq\epsilon\},\ \rho_1,\rho_2\in U_\infty$$

$$\sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}<\infty,\ \mu=||f||_{S^p},\ \frac{1}{\mu(T,\rho_1)}\int_{M(T,\epsilon,f)}\rho_2(t)<\frac{\epsilon}{\mu+1}.$$

We obtain

$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}||\psi(\theta)||\Big)dt$$

$$\begin{aligned}
&\leq \frac{\tilde{M}}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta-k}^{\theta-k+1} ||f(s)||^p ds\Big)^{\frac{1}{p}}\Big)dt. \\
&\leq \frac{\tilde{M}}{\mu(T,\rho_1)} \int_{M(T,\epsilon,f)} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta-k}^{\theta-k+1} ||f(s)||^p ds\Big)^{\frac{1}{p}}\Big)dt. \\
&+ \frac{\tilde{M}}{\mu(T,\rho_1)} \int_{[-T,T]/M(T,\epsilon,f)} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta-k}^{\theta-k+1} ||f(s)||^p ds\Big)^{\frac{1}{p}}\Big)dt. \\
&\leq \frac{\tilde{M}\mu}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t)dt \\
&+ \frac{\tilde{M}}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_{\theta-k}^{\theta-k+1} ||f(s)||^p ds\Big)^{\frac{1}{p}}\Big)dt. \\
&\leq \tilde{M}\mu \sup_{T>0} \frac{\mu(t,\rho_2)}{\mu(t,\rho_1)} + \tilde{M}\epsilon.
\end{aligned}$$

Hence

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} ||\psi(\theta)||\Big)dt = 0.$$

□

From the proof of Lemma (3.1), we cn easily obtain the following Lemma.

Lemma 3.2. *We assume that the hypothesis* $(H_2), (H_4)$ *is satisfied. If* $g \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2), \rho_1, \rho_2 \in U_\infty$. *Then*

$$(\Gamma_2 g)(t) = \int_{-\infty}^{t} T(t-s)g(s)ds \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2).$$

Now we give our main theorem.

Theorem 3.3. *Assume that the hypotheses* $(H_3) - (H_6)$ *hold, then the equation* (1.1) *has a unique weighted pseudo periodic mild solution if* $(\alpha + \eta)K_f\bar{\omega} < 1$.

Proof. Define the operator $\Gamma_1 : WPAP(X) \to WPAP(X)$ by $x(t) = -f(t, Bx_t) + (\Gamma_1)x(t) + (\Gamma_2)x(t)$,
where $(\Gamma_1 x)(t) = -\int_{-\infty}^{t} AT(t-s)f(s, Bx_s)$, $(\Gamma_2)x(t) = \int_{-\infty}^{t} T(t-s)g(s, Cx_s)ds$.
If $x \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$, by the composition theorem of weight Stepanov pseudo almost periodic [Theorem 2.10 and Corollary 3.1, 3.2]

then $f(s,x_s),\ g(s,x_s) \in S^pWPAP(\mathbb{R}\times X, \mathfrak{B}, r, \rho_1, \rho_2)$. It follows that the operator defined by $(\Gamma x)(t)$ maps $WPAP$ into $WPAP$.
The linear operator $B, C \in B(Y,X)$ with $\max(||B||_{B(Y,X)}, ||C||_{B(Y,X)}) = \bar{\omega}$.
For all $x, y \in S^pWPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ and (H_2), (H_3) we have

$$\begin{aligned} ||f(t, Bx_t - f(t, By_t)||_Y &\leq K_f||Bx_t - By_t||_Y ds \\ &\leq K_f\bar{\omega}||x_t - y_t||_Y \\ &\leq K_f\bar{\omega}||x_t - y_t||_{\infty,Y}. \end{aligned}$$

Then Γ maps $WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ into itself. Hence Γ is well defined.

For $x, y \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ and for convenience, we separate two part as following:

$$\begin{aligned} ||(\Gamma_1 x)(t) - (\Gamma_1 y)(t)|| &= ||\int_{-\infty}^t AT(t-s)f(s, Bx_s) + \int_{-\infty}^t AT(t-s)f(s, By_s)ds \\ &= ||\int_{-\infty}^t AT(t-s)[f(t-s, Bx_{t-s}) - f(t-s, By_{t-s})]ds|| \\ &\leq \sum_{k=1}^{\infty}\Big(\int_{k-1}^k e^{-\omega qs}\gamma(s)\Big)^{\frac{1}{q}}\cdot\Big(\int_{k-1}^k ||f(s, Bx_s) - f(s, By_s)||^{\frac{1}{p}}\Big) \\ &\leq ||f(t+k-2+\cdot, Bx_{t+k-2+\cdot}) - f(t+k-2+\cdot, By_{t+k-2+\cdot})||_p \\ &\leq \eta K_f\bar{\omega}\cdot||x_{t+k-2+\cdot} - y_{t+k-2+\cdot}|| \\ &\leq \eta K_f\bar{\omega}\cdot||x(t+k-2+\cdot) - y(t+k-2+\cdot)|| \\ &= \eta K_f\bar{\omega}\cdot||x-y||. \end{aligned}$$

Similarly, we get

$$\begin{aligned} ||(\Gamma_2 x)(t) - (\Gamma_2 y)(t)|| &= ||\int_{-\infty}^t T(t-s)g(s, By_s) + \int_{-\infty}^t T(t-s)g(s, Cx_s)ds \\ &= ||\int_{-\infty}^t T(g(t-s, Bx_{t-s}) - g(t-s, By_{t-s})ds|| \\ &\leq \sum_{k=1}^{\infty}\big(\int_{k-1}^k e^{-\omega qs}\Big)^{\frac{1}{q}}\cdot\big(\int_{k-1}^k ||g(s, Bx_s) - g(s, By_s)||^{\frac{1}{p}}\Big) \\ &\leq \alpha K_f\bar{\omega}\cdot||x-y||. \end{aligned}$$

Hence, for any $x, y \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ we obtain

$$||(\Gamma x)(t) - (\Gamma y)(t)||_{\infty,Y} \leq (\alpha + \eta) K_f \bar{\omega} \cdot ||x - y||_{\infty,Y}.$$

Then Γ is a contraction map if $(\alpha + \eta) K_f \bar{\omega} < 1$.
Therefore, Γ has a unique fixed point $x \in WPAP(\mathbb{R}, X, r, \rho_1, \rho_2)$ such that $\Gamma x = x$. This function u is a Stepanov weighted pseudo almost periodic mild solution of equation(1.1).
This completes the proof of Theorem. □

4. Examples and Applications

In this section we consider a simple application of our abstracts results, we give an example as follows $\frac{\partial}{\partial t}\big[u(t,\xi)+\int_{-\infty}^{t}\int_0^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds\big]$

$$= \begin{cases} \frac{\partial^2}{\partial \xi^2} u(t,\xi) + g\big(t, q(\xi)u_t\big), \ t \geq 0, 0 \leq \xi \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \ t \geq 0, \\ u(\theta,\xi) = \phi(\theta,\xi) = 0, \ \theta \leq 0, \ 0 \leq \xi \leq \pi. \end{cases} \tag{1.2}$$

Where the functions a_0, a, a_1, b, and ϕ satisfy appropriate conditions. To abstract this problem we shall take $(X, ||\cdot||) = L^2(0,\pi)$ and define $x(t) = u(t,\cdot)$. the operator A is given by

$$Af(\nu) = f^{''}(\nu)$$

with domain

$$D(A) = \{f(\cdot) \in L^2([0,\pi]) : f^{''}(\cdot) \in L^2([0,\pi]), \ f(0) = f(\pi) = 0\}.$$

A is the infinetesimal generator of a C_0-semigroup $T(t)$ on $L^2[0,\phi]$ with

$$||T(t)|| \leq e^{-t}, t \geq 0.$$

Define the function $f : \mathbb{R} \times \mathfrak{B} \to X$ by

$$f(t,\psi)(\xi) = \int_{-\infty}^{t}\int_0^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds,$$

then the above partial differential equation can be rewritten as an abstract system of the equation (1.1) with $u(t) = u(t,\cdot)$.

By the main results one can easily show that the equation (1.2) has a unique stepanov weighted pseudo almost periodic mild solution. For more details examples, we refer to [7],[12], [19] and references therein.

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